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The τ-Value, The Core and Semiconvex Games

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Abstract: The τ-value for cooperative n-person games is central in this paper. Conditions are given which guarantee that the τ-value lies in the core of the game. A full-dimensional cone of semiconvex games is introduced. This cone contains the cones of convex and exact games and there is a simple formula for the τ-value for such games. The subclass of semiconvex games with constant gap function is characterized in several ways. It turns out to be an $(n + 1)$-dimensional cone and for all games in this cone the Shapley value, the nucleolus and the τ-value coincide.

1 Introduction and Summary

This paper deals with n-person games in characteristic function form. Let N be the set $\{1, 2, \ldots, n\}$ $(n \geqslant 2)$ and let 2^N be the family of all subsets of N. A function $v : 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$ is called an *n-person game* (*in characteristic function form*). In this context elements of N are called *players* and elements of 2^N *coalitions*. The number $v(S)$ is called the *worth* of coalition S in the game v. The family of n-person games is denoted by G^n. Special attention will be paid to the τ-value τ^v for an n-person game v, introduced in *Tijs*, and to a subclass of G^n, consisting of the semiconvex games.

In section 2 we recall the definition of the τ-value for games belonging to a specific subclass Q^n of G^n. We also recall a calculation method for the τ-value τ^v of games $v \in Q^n$. The theorems 2.2 and 2.3, which will be used in the sections 4 and 5, are restatements of results established in other papers dealing with the τ-value. In theorem 2.4 a necessary and sufficient condition is given for the τ-value to lie in the core of the game.

In section 3 we define the maximal remainder map for n-person games, and some interesting properties are collected in theorems 3.1, 3.2 and 3.3. In theorem 3.4 it is shown that the τ-value lies in the core of a game if and only if the τ-value is a fixed point of the maximal remainder map.

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In section 4 we introduce the subclass SC^n of G^n of semiconvex games. We show that this class is a full-dimensional cone in Q^n, which contains the convex games and also the exact games (theorems 4.6, 4.7 (i) and 4.4). In theorem 4.7 (iii) a simple formula for the τ-value of a semiconvex game is given, and a necessary and sufficient condition for the τ-value to belong to the core. Some results for semiconvex games with a small number of players are given in theorems 4.8 and 4.9.

In section 5 the class EG^n of games with a non-negative constant gap function is considered. The main characterization of this subclass of SC^n is given in theorem 5.2, while a geometric characterization is given in theorem 5.5. Several other characterizations are listed in proposition 5.6. Furthermore it is shown in theorem 5.3 that for the class EG^n the τ-value coincides with the nucleolus [cf. *Schmeidler*, 1969; *Kohlberg*] and the Shapley value [cf. *Shapley*, 1953].

In the final section three illustrative examples are given.

We conclude this section with two well-known definitions and some notation. A game $v \in G^n$ is called *superadditive* (s.a.) if

$$v(S \cup T) \geqslant v(S) + v(T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset. \tag{1.1}$$

A game $v \in G^n$ is called *zero-normalized* if

$$v(\{i\}) = 0 \text{ for all } i \in N.$$

In this paper the following notation is used. For a vector $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $S \in 2^N$ we often write $x(S)$ instead of $\sum_{i \in S} x_i$, where $x(\emptyset) = 0$. We also write (i), (ij), (ijk), $N - i$, $S \cup i$ instead of $\{i\}$, $\{i, j\}$, $\{i, j, k\}$, $N - \{i\}$, $S \cup \{i\}$ etc. For $S \in 2^N$ the number of players in S is denoted by $|S|$. The nucleolus of a game v is denoted by $n(v)$ and the Shapley value by $\phi(v)$.

2. The τ-value

We start by recalling the definition of the τ-value. In *Tijs*, the τ-value is defined for a subclass of n-person games as the feasible compromise between two vectors, namely the upper vector and the lower vector of the game. The *upper vector* $b^v \in \mathbf{R}^n$ of a game $v \in G^n$ is defined by

$$b_i^v = v(N) - v(N - \{i\}, \quad \text{for all } i \in N. \tag{2.1}$$

The i-th coordinate b_i^v of b^v is called the *marginal contribution of player i* in the game v.

Let $v \in G^n$. To define the lower vector $a^v \in \mathbf{R}^n$ of the game v, we first introduce for each $i \in N$ and each $S \in 2^N$ with $i \in S$ the expression

$$R_i^v(S) = v(S) - \sum_{k \in S-\{i\}} b_k^v = v(S) - b^v(S - \{i\}) \tag{2.2}$$

which is called the *remainder for player i in the coalition S.* Note that $R_i^v(S)$ is the amount which remains for player i if the players in coalition S divide their worth $v(S)$ in such a way that each player in S, except player i, obtains his marginal contribution. Now the i-th *coordinate* a_i^v *of the lower vector* is defined by

$$a_i^v = \max_{S:i \in S} R_i^v(S). \tag{2.3}$$

So a_i^v is the greatest possible remainder for player i.

The terms upper and lower vector are explained by the next proposition, which states that the upper (lower) vector of $v \in G^n$ is an upper (lower) bound for the core $C(v)$ of the game v, where

$$C(v) = \{x \in \mathbf{R}^n; x(N) = v(N) \text{ and } x(S) \geqslant v(S) \text{ for all } S \in 2^N\}. \tag{2.4}$$

Proposition 2.1

Let $v \in G^n$. Then

$$a_i^v \leqslant x_i \leqslant b_i^v \text{ for all } i \in N \text{ and each } x \in C(v). \tag{2.5}$$

Proof

Let $x \in C(v)$. By (2.4) and (2.1), we have for all $i \in N$ $x_i = x(N) - x(N-i) =$ $= v(N) - x(N-i) \leqslant v(N) - v(N-i) = b_i^v$. From this and (2.4), it follows that $R_j^v(S) = v(S) - b^v(S-j) \leqslant v(S) - x(S-j) \leqslant x_j$ for all $j \in N$ and each $S \in 2^N$ with $j \in S$. Hence, by (2.3), $a_j^v \leqslant x_j$ for all $j \in N$. □

In *Tijs/Lipperts* it is shown that the inequalities in (2.5) are sharp for large classes of games, e.g., the class of convex games.

Let

$$Q^n = \{v \in G^n : a^v \leqslant b^v \text{ and } a^v(N) \leqslant v(N) \leqslant b^v(N)\}. \tag{2.6}$$

Thus, Q^n consists of those n-person games where the lower vector lies below the upper vector and the sum of the coordinates of the lower (upper) vector is not greater (not smaller) than the worth of the grand coalition. For each game $v \in Q^n$

the *τ-value* τ^v is defined as the unique point on the line segment $[a^v, b^v]$ with end points a^v and b^v, for which the sum of the coordinates is equal to $v(N)$.

Driessen/Tijs [1983] gave a calculation method for the τ-value of a game $v \in Q^n$, which avoids the calculation of the lower vector a^v. For that purpose they define for any $v \in G^n$ the map $g^v : 2^N \to \mathbf{R}$ and the vector $\lambda^v \in \mathbf{R}^n$ as follows:

$$g^v(S) = \sum_{i \in S} b_i^v - v(S) = b^v(S) - v(S) \text{ for all } S \in 2^N, \tag{2.7}$$

$$\lambda_i^v = \min_{S: i \in S} g^v(S) \text{ for all } i \in N. \tag{2.8}$$

The map g^v is called the *gap function* of the game v and $g^v(S)$ the *gap of coalition S* in v. It follows immediately from the definitions (2.2), (2.3), (2.7) and (2.8) that $\lambda^v = b^v - a^v$. In view of this and proposition 2.1, the vector λ^v is called the *concession vector* of the game v. Furthermore, the class Q^n can be rewritten as

$$Q^n = \{v \in G^n : g^v(S) \geqslant 0 \text{ for all } S \in 2^N - \{\emptyset\} \text{ and } \lambda^v(N) \geqslant g^v(N)\}. \tag{2.9}$$

In the next theorem a formula for the τ-value τ^v of a game $v \in Q^n$ is given in terms of b^v, λ^v and g^v. Observe that the formula for the τ-value in theorem 2.2 (ii) is similar to the formulas proposed by the Tennessee Valley Authority (TVA) to solve the problem of fair allocation of joint costs. This problem was considered by the TVA in the 1930's to apportion costs of dam systems among participatory uses [*Straffin/Heaney*].

Theorem 2.2

[*Driessen/Tijs*, 1983]. Let $v \in Q^n$.

(i) If $g^v(N) = 0$, then $\tau^v = b^v$ and $C(v) = \{b^v\}$.

(ii) If $g^v(N) > 0$, then $\tau^v = b^v - g^v(N)(\lambda^v(N))^{-1}\lambda^v$.

Tijs/Lipperts proved that the class Q^n is a $(2^n - 1)$-dimensional cone in the $(2^n - 1)$-dimensional linear space G^n. Note that Q^n contains each game with non-empty core because of (2.5) and (2.6). The τ-value on Q^n satisfies the standard desirable properties for values; namely, individual rationality, efficiency, symmetry, dummy player property, S-equivalence property, and continuity property [cf. *Tijs*]. An extension of the τ-value from Q^n to the space G^n of all n-person games is given in *Driessen/Tijs* [1984 a].

In section 5 we will characterize the class EG^n of games for which the gap function is non-negative and constant, with the aid of the $(2^n - 1)$-dimensional subcone

$\tilde{Q}^n$ of Q^n considered in *Driessen/Tijs* [1983]. This class $\tilde{Q}^n$ is the subclass of Q^n, consisting of those games where the gap function attains its minimum at N, i.e.

$$\tilde{Q}^n = \{v \in Q^n : g^v(N) \leqslant g^v(S) \text{ for all } S \in 2^N - \{\emptyset\}\}. \tag{2.10}$$

Some interesting facts for this subclass are collected in the next theorem.

Theorem 2.3

[*Driessen/Tijs*, 1983]. Let $v \in \tilde{Q}^n$. Then

(i) $\tau^v = b^v - n^{-1} g^v(N)\, 1_n$ where $1_n := (1, 1, \ldots, 1) \in \mathbf{R}^n$.

(ii) $C(v)$ is the convex hull of n points $f^1, f^2, \ldots, f^n$ where $f^i := b^v - g^v(N)\, e^i = (b^v_1, \ldots, b^v_{i-1}, b^v_i - g^v(N), b^v_{i+1}, \ldots, b^v_n)$.

(iii) the τ-value of v is the center of gravity of the extreme points of the core, i.e.,
$$\tau^v = n^{-1} \sum_{i=1}^{n} f^i.$$

(iv) the τ-value τ^v of v is equal to the nucleolus $n(v)$ of v.

(v) the restriction of τ to $\tilde{Q}^n$ is additive, i.e. $\tau^{v+w} = \tau^v + \tau^w$ for all $v, w \in \tilde{Q}^n$.

An interesting question is whether the τ-value lies in the core. If $g^v(N) = 0$, then $\tau^v \in C(v)$ by theorem 2.2 (i). If $g^v(N) > 0$, then not necessarily is $\tau^v \in C(v)$, as example 6.2 in section 6 will show. In the next theorem and in section 3, theorem 3.4 (ii), we give necessary and sufficient conditions for $\tau^v \in C(v)$.

Theorem 2.4

Let $v \in Q^n$ with $g^v(N) > 0$. Then $\tau^v \in C(v)$ iff $(g^v(N))^{-1} \lambda^v(N) \geqslant (g^v(S))^{-1} \lambda^v(S)$ for all $S \in 2^N$ with $1 < |S| < n-1$ and $g^v(S) \neq 0$.

Proof

By the definition of the τ-value we have $\tau^v(N) = v(N)$ and $a^v_i \leqslant \tau^v_i \leqslant b^v_i$ for all $i \in N$. Hence, we have for all $i \in N$ that $\tau^v_i \geqslant a^v_i \geqslant R^v_i(i) = v(i)$ because of (2.3) and (2.2); in addition $\tau^v(N-i) = \tau^v(N) - \tau^v_i = v(N) - \tau^v_i \geqslant v(N) - b^v_i = v(N-i)$ because of (2.1). It follows from (2.4) that $\tau^v \in C(v)$ iff

$$\tau^v(S) \geqslant v(S) \text{ for all } S \in 2^N \text{ with } 1 < |S| < n-1. \tag{2.11}$$

But (2.11) is, in view of theorem 2.2 (ii), equivalent to

$$g^v(S) \geqslant g^v(N)(\lambda^v(N))^{-1}\lambda^v(S) \text{ for all } S \in 2^N \text{ with } 1 < |S| < n-1.$$

By noting that $g^v(S) = 0$ implies $\lambda^v(S) = 0$, the statement in the theorem follows immediately. □

From $\{S \in 2^N : 1 < |S| < n-1\} = \emptyset$ whenever $n \in \{2, 3\}$, we obtain the following

Corollary 2.5

For each $v \in Q^2$ and each $v \in Q^3$, $\tau^v \in C(v)$.

3 The Maximal Remainder Map and the Core of a Game

Inspired by the definition of the lower vector of a game and the paper of *Bennett/Wooders*, we introduce for each n-person game v a map $M^v : \mathbf{R}^n \to \mathbf{R}^n$, which can be called the *maximal remainder map* of v. In this section we study this map with respect to the core and the τ-value of a game v.

For each $x \in \mathbf{R}^n$, the i-th coordinate $M_i^v(x)$ of $M^v(x)$ is defined by

$$M_i^v(x) = \max_{S : i \in S} [v(S) - x(S - \{i\})].$$

One can interpret $v(S) - x(S - \{i\})$ as the remainder for player i in coalition S, when $v(S)$ is divided in such a way that each other player $j \in S$ obtains x_j. $M_i^v(x)$ is then the greatest possible remainder for player i. Note that

$$a^v = M^v(b^v). \tag{3.1}$$

We collect some properties of M^v in the following theorem. The proofs are straightforward and left to the reader.

Theorem 3.1

Let v be an n-person game. Then

(i) M^v is anti-monotonic, i.e., for all $x, y \in \mathbf{R}^n$ we have

$$x \leqslant y \Rightarrow M^v(x) \geqslant M^v(y).$$

(Here $x \leqslant y$ means: $x_i \leqslant y_i$ for all $i \in \{1, 2, \ldots, n\}$.)

(ii) M^v is a convex map, i.e., for all $x, y \in \mathbf{R}^n$ and $\alpha \in (0, 1)$,

$$M^v(\alpha x + (1-\alpha) y) \leqslant \alpha M^v(x) + (1-\alpha) M^v(y).$$

(iii) $M^v(x) \leqslant x$ iff no coalition can improve the payoff $x \in \mathbf{R}^n$, i.e.,

$$M^v(x) \leqslant x \text{ iff } x(S) \geqslant v(S) \text{ for all } S \in 2^N.$$

(iv) $M^v(x) \geqslant x$ iff for each $i \in N$ there exists an $S \in 2^N$ with $i \in S$ and $x(S) \leqslant v(S)$.

(v) $x(N) \leqslant v(N) \Rightarrow M^v(x) \geqslant x$.

The following theorem provides a characterization of the core elements of a game in terms of M^v.

Theorem 3.2

Let $v \in G^n$. Then $x \in \mathbf{R}^n$ is an element in the core of v iff x is feasible (i.e. $x(N) = v(N)$) and x is a fixed point of M^v (i.e., $M^v(x) = x$).

Proof

For the 'only if' part of the theorem, suppose $x \in C(v)$. Then $x(N) = v(N)$ and by theorem 3.1 (iii) and (v), $M^v(x) = x$.

Conversely, suppose $x(N) = v(N)$ and $M^v(x) = x$. Then (iii) of theorem 3.1 implies that $x \in C(v)$. □

Although the core may be empty, the map M^v always has fixed points. Fixed points of M^v are called equilibrium-reservation-prices in *Bennett/Wooders*.

Theorem 3.3

Let $v \in G^n$. Then $\{x \in \mathbf{R}^n : M^v(x) = x\} \neq \emptyset$.

Proof

Let $x^0 = (\alpha, \alpha, \ldots, \alpha)$, where α is a large real number such that $x^0(S) \geqslant v(S)$ for all $S \in 2^N$. If $N^0 = \{i \in N :$ there exists an $S \in 2^N$ with $i \in S$ and $x^0(S) = v(S)\}$ equals N, then x^0 is a fixed point of M^v in view of theorem 3.1 (iii) and (iv). If $j \in N - N^0$, define $x^1 = x^0 - \epsilon^1 e^j$, where e^j is the j-th unit vector in $\mathbf{R}^n$ and ϵ^1 is the largest positive real number such that $x^1(S) \geqslant v(S)$ for every coalition S. Then $N^1 = \{i \in N$: there is an $S \in 2^N$ with $i \in S$ and $x^1(S) = v(S)\} \overset{\supset}{\neq} N^0$ and $j \in N^1$. If $N^1 = N$, then $x^1 = M^v(x^1)$. Otherwise, we construct, similarly, x^2, etc. After at most n steps we find a fixed point of M^v. □

In (ii) of the following theorem we give the announced necessary and sufficient condition for τ^v to be a core element.

Theorem 3.4

Let $v \in Q^n$. Then

(i) $M^v (a^v) \geqslant a^v, M^v (b^v) \leqslant b^v$.

(ii) $\tau^v \in C(v)$ iff τ^v is a fixed point of M^v.

Proof

(i) Since $a^v \leqslant b^v$ for $v \in Q^n$, theorem 3.1 (i) and (3.1) imply

$M^v (a^v) \geqslant M^v (b^v) = a^v$ and $M^v (b^v) = a^v \leqslant b^v$.

(ii) is a direct consequence of theorem 3.2. □

4 Convex Games, Exact Games and Semiconvex Games

A game $v \in G^n$ is called a *convex game* [cf. *Shapley*, 1971] if

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T) \text{ for all } S, T \in 2^N. \tag{4.1}$$

The set of convex n-person games is denoted by C^n. *Shapley* [1971] gives several characterizations of convex games. We list some of these in the following theorem, where the real-valued function m on $2^N \times 2^N$ is defined by

$$m(S, T) = v(S \cup T) - v(S) - v(T) \text{ for all } S, T \in 2^N$$

and where the differencing operator Δ_S from the set F_n of real-valued functions on 2^N into itself is defined by

$$(\Delta_S f)(T) = f(T \cup S) - f(T - S) \text{ for all } f \in F_n \text{ and } S, T \in 2^N.$$

Theorem 4.1

[*Shapley*, 1971]. Let $v \in G^n$. The following assertions are equivalent.

(i) v is a convex game.

(ii) $m(S_1, T) \leqslant m(S_2, T)$ for all $S_1, S_2, T \in 2^N$ with $S_1 \subset S_2 \subset N - T$.

(iii) $\Delta_Q (\Delta_R v)(S) \geqslant 0$ for all $Q, R, S \in 2^N$.

(iv) $v(S_1 \cup \{i\}) - v(S_1) \leqslant v(S_2 \cup \{i\}) - v(S_2)$ for all $S_1, S_2 \in 2^N$ and $i \in N$ with $S_1 \subset S_2 \subset N - \{i\}$.

For other properties of convex games we refer to *Shapley* [1971].

A class, larger than the class of convex n-person games, is the class E^n of exact n-person games [cf. *Schmeidler*, 1972]. $v \in G^n$ is called an *exact game* if

$$\text{for each } S \in 2^N \text{ there is an } x \in C(v) \text{ with } x(S) = v(S). \tag{4.2}$$

The gap function is increasing for exact games as the following theorem shows.

Theorem 4.2

Let $v \in E^n$ and $\emptyset \neq S \subset T \subset N$. Then $g^v(S) \leqslant g^v(T)$.

Proof

It is sufficient to show that for each $S \in 2^N - \{\emptyset\}$ and each $i \in N - S$, $g^v(S) \leqslant g^v(S \cup i)$. Take such an S and i. Since v is exact, there is an $x \in C(v)$ with $x(S) = v(S)$. Since $x \in C(v)$ we have $x(S \cup i) \geqslant v(S \cup i)$ and by (2.5) also $b_i^v \geqslant x_i$. Then $g^v(S \cup i) = b^v(S \cup i) - v(S \cup i) = b^v(S) + b_i^v - v(S \cup i) \geqslant b^v(S) + x_i - x(S \cup i) = b^v(S) - x(S) = b^v(S) - v(S) = g^v(S)$. □

Now we introduce a new class SC^n of games, the elements of which we call *semiconvex games.*

Definition 4.3

Let $v \in G^n$. The game v is called *semiconvex* if v is s.a. and if for all $i \in N$ and all $S \in 2^N$ with $i \in S$

$$g^v(\{i\}) \leqslant g^v(S). \tag{4.3}$$

Note that the gap function g^v of a semiconvex game v is non-negative because for any $i \in N$, $g^v(\{i\}) = b_i^v - v(\{i\}) = v(N) - v(N - \{i\}) - v(\{i\}) \geqslant 0$ by the superadditivity of v.

That convex games are semiconvex follows from

Theorem 4.4

$C^n \subset E^n \subset SC^n$.

Proof

(i) *Schmeidler* [1972] noted that every convex game is exact. This becomes clear since for $v \in C^n$ and, e.g., $S = \{1, 2, \ldots, s\}$, the vector $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ defined by

$$x_i = v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i - 1\}) \text{ for all } i \in N$$

is a core-element of v satisfying $x(S) = v(S)$.

(ii) Let $v \in E^n$. Then inequality (4.3) holds for v because of theorem 4.2. It remains to show that v is s.a. Let $S, T \in 2^N$ with $S \cap T = \emptyset$. By (4.2), there is an $y \in C(v)$ with $y(S \cup T) = v(S \cup T)$. Then $v(S \cup T) = y(S \cup T) = y(S) + y(T) \geq v(S) + v(T)$. So, v is s.a. and hence, $E^n \subset SC^n$. □

It is easy to check that $C^2 = SC^2$ and $C^3 = SC^3$. We give an example of a semiconvex 4-person game, which is not convex nor exact.

Example 4.5

Let v be the s.a. 4-person game with

$v(S) = 0, 1, 1, 2$ if $|S| = 1, 2, 3, 4$, respectively.

Then $b^v = (1, 1, 1, 1), g^v(S) = 1, 1, 2, 2$ if $|S| = 1, 2, 3, 4$, respectively; so v satisfies (4.3). Thus $v \in SC^4$. Because $v(123) + v(1) < v(12) + v(13)$, $v \notin C^4$. Also $v \notin E^4$: there is no $x \in C(v)$ with $x_1 = 0 = v(1)$, because $C(v) = \{(1/2, 1/2, 1/2, 1/2)\}$.

It is well-known that C^n is a $(2^n - 1)$-dimensional cone. It is easy to see from (4.3) and (1.1) that SC^n is also a cone, which is in view of theorem 4.4 also full-dimensional in the $(2^n - 1)$-dimensional linear space G^n. Hence

Theorem 4.6

SC^n is a $(2^n - 1)$-dimensional cone in G^n.

For further use we note that for $k \in (0, \infty)$ and $c \in \mathbf{R}^n$ we have

$$v \in SC^n \Leftrightarrow kv + c \in SC^n. \tag{4.4}$$

(Here $(kv + c)(S) = kv(S) + \sum_{i \in S} c_i$ for each $S \in 2^N$.)

In the next theorem we give a simple formula for the τ-value for semiconvex games and also an easy criterion to verify whether the τ-value lies in the core for a zero-normalized semiconvex game.

Theorem 4.7

Let $v \in SC^n$. Then

(i) $v \in Q^n$.

(ii) $a^v = (v(\{1\}), v(\{2\}), \ldots, v(\{n\}))$.

(iii) If v is also zero-normalized and $b^v(N) > 0$, then

$\tau^v = (b^v(N))^{-1} v(N) b^v$ and

$\tau^v \in C(v)$ iff $(g^v(N))^{-1} b^v(N) \geq (g^v(S))^{-1} b^v(S)$ for all $S \in 2^N$ with $1 < |S| < n - 1$ and $g^v(S) \neq 0$.

Proof

$v \in SC^n$ implies that $\lambda_i^v = g^v(i)$ for all $i \in N$.

(i) We have $\lambda^v(N) = \sum_{i \in N} \lambda_i^v = \sum_{i \in N} g^v(i) = \sum_{i \in N} b_i^v - \sum_{i \in N} v(i) \geq b^v(N) - v(N) = g^v(N)$, where the inequality follows from the s.a. of v. We noted that $v \in SC^n$ also implies $g^v \geq 0$. Hence, by (2.9), $v \in Q^n$.

(ii) We always have that $\lambda^v = b^v - a^v$. Thus, $a_i^v = b_i^v - \lambda_i^v = b_i^v - g^v(i) = v(i)$ for all $i \in N$.

(iii) Let v be zero-normalized and $b^v(N) > 0$. Then by (ii), $a_i^v = v(i) = 0$ for all $i \in N$. Thus $\tau^v = \alpha\, b^v$ where $\alpha \in [0, 1]$ is chosen such that $\tau^v(N) = v(N)$. This implies that $\alpha = (b^v(N))^{-1} v(N)$ and hence $\tau^v = (b^v(N))^{-1} v(N)\, b^v$. Note that for any $S \in 2^N - \{\emptyset\}$, we have $\lambda^v(S) = b^v(S) - a^v(S) = b^v(S)$. In particular, $\lambda^v(N) = b^v(N) > 0$, which implies by (2.8) that $g^v(N) > 0$. The second statement of (iii) now follows immediately from theorem 2.4. □

We note that by theorem 4.7 (iii), the distribution proposed by the τ-value for a semiconvex game is equal to the allocation proposed by the "alternative cost avoided" method in *Straffin/Heaney* [page 40].

In the remainder of this section we discuss some results for semiconvex games when the number of players is small. First, we note that for each semiconvex game with 2 or 3 players $\tau^v \in C(v)$, by corollary 2.5. It is not difficult to find a semiconvex 5-person game v with $\tau^v \notin C(v)$. But for semiconvex 4-person games the core is non-empty and τ^v is an element of the core as we see in the following theorem.

Theorem 4.8

Let $v \in SC^4$. Then $C(v) \neq \emptyset$ and $\tau^v \in C(v)$.

Proof

We only have to prove that $\tau^v \in C(v)$. In view of (4.4) we will suppose w.l.o.g. that v is zero-normalized, so $v(i) = 0$ for all $i \in N$. It follows from the s.a. of v that $v(S) \geq \sum_{i \in S} v(i) = 0$ for all $S \in 2^N - \{\emptyset\}$. In the case $g^v(N) = 0$ we have that $\tau^v \in C(v)$ by theorems 4.7 (i) and 2.2 (i). Consider the case $g^v(N) > 0$. Then $b^v(N) > 0$ because $v(N) \geq 0$. In view of theorem 4.7 (iii), it is sufficient to show that

$$g^v(S)\, b^v(N) \geq g^v(N)\, b^v(S) \text{ for all } S \in 2^N \text{ with } |S| = 2. \tag{4.5}$$

But (4.5) is equivalent to

$$b^v(S)\, v(N) \geq b^v(N)\, v(S) \text{ for all } S \in 2^N \text{ with } |S| = 2$$

or to

$$b^v(S)\,[v(N)-v(S)] \geqslant b^v(N-S)\,v(S) \text{ for all } S \in 2^N \text{ with } |S| = 2. \tag{4.6}$$

Let $S = \{i_1, i_2\}, N-S = \{j_1, j_2\}$. We have to show that (4.6) holds. Note that

$$b^v(S) = b^v_{i_1} + b^v_{i_2} = g^v(i_1) + g^v(i_2) \leqslant 2\,g^v(S) = 2\,b^v(S) - 2\,v(S) \tag{4.7}$$

where the inequality follows from (4.3) since $v \in SC^4$. But (4.7) implies

$$b^v(S) \geqslant 2\,v(S). \tag{4.8}$$

Since v is s.a. and zero-normalized we have for j_r $(r=1,2)$: $v(S) \leqslant v(N-j_r)$ and $v(S) \leqslant v(N)$. Then

$$b^v(N-S) = b^v_{j_1} + b^v_{j_2} = \sum_{r=1}^{2} [v(N) - v(N-j_r)] \leqslant 2\,[v(N) - v(S)]. \tag{4.9}$$

Because $v(S) \geqslant 0$ and $v(N) - v(S) \geqslant 0$, we conclude from (4.8) and (4.9) that (4.6) holds. □

In *Tijs*, theorem 4.5, it is shown that for each $v \in Q^2$

$$\tau^v = \phi(v) = n(v) = 1/2\,(v(12) + v(1) - v(2),\, v(12) - v(1) + v(2)) \in C(v).$$

For 3-person games $v \in Q^3$ the τ-value, Shapley value and nucleolus no longer coincide in general. But it follows from corollary 5.4 in the next section that for semiconvex 3-person games with $g^v(N) = 0$ the three solution concepts do coincide. For games $v \in SC^3$ with $g^v(N) > 0$ we give conditions in the next theorem which guarantee that the solution concepts coincide. The following equalities play a role:

$$v(1) + v(23) = v(2) + v(13) = v(3) + v(12) \tag{4.10}$$

$$v(12) + v(13) + v(23) = v(N) + \sum_{i=1}^{3} v(i) \tag{4.11}$$

Theorem 4.9

Let $v \in SC^3$ with $g^v(N) > 0$. Then

(i) $\tau^v = \phi(v)$ iff (4.10) or (4.11) holds.

(ii) $\tau^v = n(v)$ iff (4.10) or (4.11) holds.

(iii) If (4.10) holds, then

$$\tau^v_i = \phi_i(v) = n_i(v) = v(\{i\}) + \frac{1}{3}\,[v(N) - \sum_{i=1}^{3} v(\{i\})] \text{ for all } i \in N.$$

(iv) If (4.11) holds, then

$$\tau_i^\nu = \phi_i(\nu) = n_i(\nu) = 1/2\,[b_i^\nu + \nu(\{i\})] \text{ for all } i \in N.$$

Proof

W.l.o.g. we suppose that ν is zero-normalized. Together with $g^\nu(N) > 0$ and the s.a. of ν, this implies that $b^\nu(N) > \nu(N) \geqslant \sum_{i \in N} \nu(i) = 0$. Thus, by theorem 4.7 (iii), $\tau^\nu = (b^\nu(N))^{-1}\,\nu(N)\,b^\nu$.

(i) Put $N = \{i, j, k\}$. Then

$$\phi_i(\nu) := (n!)^{-1} \sum_{S: i \notin S} |S|!\,(n - |S| - 1)!\,[\nu(S \cup i) - \nu(S)] =$$

$$= \frac{1}{6}\nu(ij) + \frac{1}{6}\nu(ik) + \frac{1}{3}b_i^\nu = -\frac{1}{6}b^\nu(N) + \frac{1}{3}\nu(N) + \frac{1}{2}b_i^\nu.$$

It follows that

$$\phi_i(\nu) = \tau_i^\nu \text{ iff } \left(\frac{1}{3}b^\nu(N) - b_i^\nu\right)(2\nu(N) - b^\nu(N)) = 0.$$

Hence

$$\phi(\nu) = \tau^\nu \text{ iff } b_1^\nu = b_2^\nu = b_3^\nu \text{ or } 2\nu(N) = b^\nu(N).$$

Thus $\phi(\nu) = \tau^\nu$ iff (4.10) or (4.11) holds.

(ii) We have

$$\begin{aligned} e(S, \tau^\nu) := \nu(S) - \tau^\nu(S) &= 0 && \text{if } S = N, \emptyset \\ &= -(b^\nu(N))^{-1}\,\nu(N)\,b_i^\nu && \text{if } S = \{i\} \\ &= -(b^\nu(N))^{-1}\,g^\nu(N)\,b_i^\nu && \text{if } S = N - \{i\}. \end{aligned}$$

If (4.10) holds, we have $b_1^\nu = b_2^\nu = b_3^\nu$ and hence $e(1, \tau^\nu) = e(2, \tau^\nu) = e(3, \tau^\nu)$, $e(12, \tau^\nu) = e(13, \tau^\nu) = e(23, \tau^\nu)$. If (4.11) holds, we have $g^\nu(N) = \nu(N)$ and hence $e(1, \tau^\nu) = e(23, \tau^\nu)$, $e(2, \tau^\nu) = e(13, \tau^\nu)$, $e(3, \tau^\nu) = e(12, \tau^\nu)$. Since the sets $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1,2\}, \{1,3\}, \{2,3\}\}$, $\{\{i\}, N - \{i\}\}$ with $i \in N$, are all balanced, we conclude that $\tau^\nu = n(\nu)$ if (4.10) or (4.11) holds [cf. *Kohlberg*].

Now we outline the proof for the converse statement of (ii). Suppose $\tau^\nu = n(\nu)$. We know that $b_i^\nu \geqslant 0$ for all $i \in N$ and $b^\nu(N) > 0$. The s.a. of ν and $g^\nu(N) > 0$ imply that at most one coordinate of b^ν is equal to zero. In the case that one coordinate of b^v is equal to zero, we have by s.a. and (4.3) that $2\,v(N) = b^v(N)$, i.e., (4.11) holds (without using $\tau^v = n(v)$). There remains the case where all coordinates of b^v are positive. If $v(N) = 0$, then $v(S) = 0$ for all $S \in 2^N$ by s.a. of v, and (4.10) and (4.11) hold trivially. Next let $v(N) > 0$. Then $e(S, \tau^v) < 0$ for all $S \neq \emptyset, N$. If $g^v(N) = v(N)$, then (4.11) holds. So let $g^v(N) \neq v(N)$, which implies that $e(i, \tau^v) \neq e(N-i, \tau^v)$ for all $i \in N$. Since $\tau_i^v > 0 = v(i)$ for all $i \in N$ and $\tau^v = n(v)$, the set $W = \{T : T \in 2^N \text{ and } e(T, \tau^v) = \max_{S \neq \emptyset, N} e(S, \tau^v)\}$ must be balanced [cf. *Kohlberg*].

It follows that $W = \{\{1\}, \{2\}, \{3\}\}$ or $W = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. But this implies that $b_1^v = b_2^v = b_3^v$. So (4.10) holds. This completes the proof of (ii).

(iii) If (4.10) holds, then $b_1^v = b_2^v = b_3^v$. Thus for all $i \in N$ $\tau_i^v = (b^v(N))^{-1}\, v(N)\, b_i^v = \frac{1}{3}\, v(N)$ and $\tau_i^v = \phi_i(v) = n_i(v)$ by (i) and (ii).

(iv) If (4.11) holds, then $2\,v(N) = b^v(N)$. So for all $i \in N$ $\tau_i^v = (b^v(N))^{-1}\, v(N)\, b_i^v = \frac{1}{2}\, b_i^v$. □

For zero-normalized semiconvex 3-person games with $g^v(N) > 0$, theorem 4.9 implies that the τ-value, the nucleolus and the Shapley value coincide if and only if the game is symmetrical or the τ-value lies at the center of the line segment $[a^v, b^v] = [0, b^v]$.

5 Games with Constant Gaps

In this section we consider the subclass EG^n of G^n consisting of those n-person games where the gap function is non-negative and constant. Hence,

$$EG^n := \{v \in G^n : g^v(S) = g^v(N) \geqslant 0 \text{ for all } S \in 2^N - \{\emptyset\}\}.$$

In the next proposition we characterize the games with constant gap functions. With the aid of that proposition, we will show that the class EG^n consists of those convex n-person games which belong to the class $\tilde{Q}^n$ defined by (2.10). By theorem 4.4, this implies that $EG^n \subset SC^n$.

Proposition 5.1

Let $v \in G^n$. Then $g^v(T) = g^v(N)$ for all $T \in 2^N - \{\emptyset\}$ iff $v(S \cup \{j\}) - v(S) = b_j^v$ for all $S \in 2^N - \{\emptyset\}$ and all $j \notin S$.

Proof

Let $S \in 2^N - \{\emptyset\}$ and $j \notin S$. Then the equality $v(S \cup j) - v(S) = b_j^v$ is equivalent to $g^v(S) = g^v(S \cup j)$. The statement in the proposition follows immediately. □

Theorem 5.2

$EG^n = C^n \cap \tilde{Q}^n$.

Proof

Let $v \in EG^n$. Then $\lambda_i^v = g^v(N)$ for all $i \in N$. So $\lambda^v(N) = n\, g^v(N) \geq g^v(N)$ since $g^v(N) \geq 0$. Thus, by (2.9), $v \in Q^n$ and hence, by (2.10), $v \in \tilde{Q}^n$. Furthermore, $v \in C^n$ since the inequality in theorem 4.1 (iv) holds for v because of proposition 5.1 and $g^v(i) \geq 0$ for all $i \in N$. Hence, $EG^n \subset C^n \cap \tilde{Q}^n$.

To prove the reverse inclusion, let $v \in C^n \cap \tilde{Q}^n$ and $S \in 2^N - \{\emptyset\}$. Because $v \in \tilde{Q}^n$, we have $g^v(S) \geq 0$ and $g^v(N) \leq g^v(S)$. By the theorems 4.4 and 4.2, we have also $g^v(S) \leq g^v(N)$. So, $g^v(N) = g^v(S)$. Hence, $v \in EG^n$. □

For games with a non-negative constant gap function several solution concepts turn out to coincide with the center of the core.

Theorem 5.3

Let $v \in EG^n$. Then $\tau^v = \phi(v) = n(v) = b^v - n^{-1} g^v(N)\, 1_n$.

Proof

By theorem 5.2, $v \in EG^n$ implies $v \in \tilde{Q}^n$. So by theorem 2.3, $\tau^v = n(v) = b^v - n^{-1} g^v(N)\, 1_n$. For the i-th coordinate of $\phi(v)$ we have in view of proposition 5.1:

$$\phi_i(v) := (n!)^{-1} \sum_{S: i \notin S} |S|!\,(n - |S| - 1)!\,[v(S \cup i) - v(S)] =$$

$$= n^{-1} v(i) + (n!)^{-1} \sum_{S: i \notin S, |S| \geq 1} |S|!\,(n - |S| - 1)!\, b_i^v =$$

$$= n^{-1} v(i) + (1 - n^{-1})\, b_i^v = b_i^v - n^{-1} g^v(i) = b_i^v - n^{-1} g^v(N).$$ □

Corollary 5.4

Let v be a semiconvex n-person game with $g^v(N) = 0$. Then $v \in EG^n$ and $\tau^v = \phi(v) = n(v) = b^v$.

Proof

In view of theorem 5.3 it is sufficient to show that $v \in EG^n$. By (4.3) we have $0 = g^v(N) \geq g^v(i) = b_i^v - v(i) \geq 0$ for all $i \in N$. Hence $b_i^v = v(i)$ for each $i \in N$. Since

$g^{v} \geqslant 0$ and v is superadditive

$$0 \leqslant v(S) - \sum_{i \in S} v(i) = v(S) - b^{v}(S) = -g^{v}(S) \leqslant 0 \text{ for all } S \in 2^{N} - \{\emptyset\}.$$

Then $g^{v}(S) = 0$ for all $S \in 2^{N} - \{\emptyset\}$, hence $v \in EG^{n}$. □

Now we give a geometric characterization of EG^{n}. For this purpose consider the family

$$A^{n} := \{v \in G^{n} : v(S) = \sum_{i \in S} v(i) \text{ for each } S \in 2^{N} - \{\emptyset\}\}$$

of additive n-person games and the game $c \in G^{n}$ with

$$c(S) := |S| - 1 \text{ for each } S \in 2^{N} - \{\emptyset\}.$$

Notice that $c \notin A^{n}$ and that A^{n} is an n-dimensional subspace of G^{n} with basis e^{1}, $e^{2}, \ldots, e^{n}$, where

$$e^{i}(S) = 1 \text{ if } i \in S, \text{ and } e^{i}(S) = 0 \text{ otherwise.}$$

Then $[c]^{+} + A^{n} := \{\alpha c + v : \alpha \geqslant 0, v \in A^{n}\}$ is an $(n+1)$-dimensional cone in G^{n}. This cone coincides with EG^{n} as the following theorem shows.

Theorem 5.5

(i) $EG^{n} = [c]^{+} + A^{n}$.

(ii) $v = g^{v}(N)\, c + \sum_{i \in N} v(i)\, e^{i}$ for each $v \in EG^{n}$.

(iii) $\dim(EG^{n}) = n + 1$.

Proof

For each $v \in A^{n}$ and $S \in 2^{N} - \{\emptyset\}$ we have $g^{v}(S) = 0$. Hence, $A^{n} \subset EG^{n}$. For each $S \in 2^{N} - \{\emptyset\}$ we have $g^{c}(S) = 1$, so $c \in EG^{n}$. Since EG^{n} is a cone, we have $[c]^{+} + A^{n} \subset EG^{n}$.

To prove the reverse inclusion, note that for each $v \in EG^{n}$ and $S \in 2^{N} - \{\emptyset\}$:

$$v(S) = b^{v}(S) - g^{v}(S) = \sum_{i \in S} g^{v}(i) + \sum_{i \in S} v(i) - g^{v}(S)$$

$$= (|S| - 1)\, g^{v}(N) + \sum_{i \in S} v(i) = g^{v}(N)\, c(S) + \sum_{i \in N} v(i)\, e^{i}(S)$$

or $v = g^v(N)\, c + \sum_{i \in N} v(i)\, e^i$, where $g^v(N) \geqslant 0$. This implies that $EG^n \subset [c]^+ + A^n$. So we have proved (i) and (ii). Now (iii) follows immediately from (i). □

Finally we mention several other characterizations of games with a constant gap function, which are similar to the characterizations of convex games (see theorem 4.1).

Proposition 5.6

Let m and Δ_S be as in section 4. For any $v \in G^n$, the following assertions are equivalent.

(i) $g^v(S) = g^v(N)$ for all $S \in 2^N - \{\emptyset\}$.

(ii) $m(S_1, T) = m(S_2, T)$ for all $S_1, S_2 \in 2^N - \{\emptyset\}$, $T \in 2^N$ with $S_1 \cap T = \emptyset$ and $S_2 \cap T = \emptyset$.

(iii) $\Delta_Q(\Delta_R\, v)(S) = 0$ for all $Q, R, S \in 2^N$ with $(S - Q) - R \neq \emptyset$.

(iv) $v(S_1 \cup S_2) + v(S_1 \cap S_2) = v(S_1) + v(S_2)$ for all $S_1, S_2 \in 2^N$ with $S_1 \cap S_2 \neq \emptyset$.

(v) $v(S \cup \{j\}) - v(S) = b_j^v$ for all $S \in 2^N - \{\emptyset\}$ and all $j \notin S$.

The proofs of the statements (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) of proposition 5.6 are straightforward and we will omit them. The statement (v) ⇔ (i) is just proposition 5.1.

6 Examples

In the first example we consider a game $v \notin EG^n$ with $\tau^v = \phi(v) = n(v)$, which shows that the class where the three solution concepts coincide is larger than EG^n.

Example 6.1

Let v be the n-person game ($n \geqslant 3$) with

$$v(S) = \Big(\sum_{i \in S} i\Big)^2 - \sum_{i \in S} i^2 \text{ for each } S \in 2^N - \{\emptyset\}.$$

This is the zero-normalized version of the game, considered in *Shapley* [1971], p. 17. v is a convex game but $v \notin EG^n$ because $g^v(N) > g^v(1, 2, \ldots, n-2)$. It is straightforward to show that $1/2\, b^v = \tau^v = \phi(v) = n(v)$.

In the next example we consider an exact 5-person game for which the three solution concepts are different.

Example 6.2

Let v be the 5-person game, described by *Rabie*, where $v(i) = 0$ for all $i \in N$, $v(1i) = v(2i) = 1$ for $i \in \{3, 4, 5\}$, $v(34) = v(35) = v(45) = 2$, $v(12) = 5$, $v(345) = 10$ and $v(S) = v(S \cap \{1, 2\}) + v(S \cap \{3, 4, 5\})$ for other $S \in 2^N$ with $|S| \geqslant 3$. This game is an exact game. Further, $b^v = (5, 5, 8, 8, 8)$. Hence, by theorem 4.7 (iii), $\tau^v = 15/(34)\, b^v$. Also, $\tau_1^v + \tau_2^v < v(12)$, so $\tau^v \notin C(v)$. This implies $\tau^v \neq n(v)$. The Shapley value $\phi(v) = (2.55, 2.55, 3.3, 3.3, 3.3) \neq \tau^v$ and also $\phi(v) \neq n(v)$ because also $\phi(v) \notin C(v)$.

In the next example we consider a semiconvex game arising from an economic situation.

Example 6.3

We consider a situation with one landlord (player 1) and $n \geqslant 2$ workers (players $2, 3, \ldots, n+1$). The total gain, if s workers are hired by the landlord, is denoted by $f(s)$. Let $N = \{1, 2, \ldots, n+1\}$. Then this situation corresponds to the $(n+1)$-person game $v: 2^N \to \mathbf{R}$ with

$$\begin{aligned} v(S) &= 0 && \text{if } 1 \notin S \\ &= f(s-1) && \text{if } 1 \in S \text{ (where } s = |S|). \end{aligned}$$

Put $\Delta = f(n) - f(n-1)$. Then $b^v = (f(n), \Delta, \ldots, \Delta) \in \mathbf{R}^{n+1}$.

In the following we assume that $f(s) \geqslant 0$ for $s \in \{1, 2, \ldots, n\}$, $f(0) = 0$ and that the marginal returns form an increasing sequence, i.e.,

$$f(s+1) - f(s) > f(s) - f(s-1) \text{ for all } s \in \{1, 2, \ldots, n-1\}. \tag{6.1}$$

For the gaps we have

(i) if $1 \notin S$, then $g^v(S) = s\,\Delta$,

(ii) $g^v(\{1\}) = f(n)$,

(iii) $g^v(N) = f(n) + n\,\Delta - f(n) = n\,\Delta$,

(iv) if $1 \in S$, then $g^v(S) = f(n) + (s-1)\,\Delta - f(s-1) \geqslant f(n)$

where the last inequality follows from (6.1).

Because $f(n) \geqslant \Delta$, we have $g^v(\{i\}) = \min_{S: i \in S} g^v(S)$ for all $i \in N$. Notice that all gaps are non-negative and that v is s.a. Hence, $v \in SC^{n+1}$. In view of theorem 4.7 (iii):

$$\tau^v = (n\,\Delta + f(n))^{-1} f(n) (f(n), \Delta, \Delta, \ldots, \Delta).$$

Chetty/Dasgupta/Raghavan calculated the nucleolus for this game. It turns out that $n(v) \neq \tau^v$. *Driessen/Tijs* [1984 b] studied the symmetrical part of the core of the game v and proved that $\tau^v \in C(v)$.

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